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Some Results on Pursuit Games for an Infinite System of Ternary Differential Equations

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Abstract

This paper aims to study a one-pursuer, one evader pursuit differential game for a higher level of infinite system that is an infinite system of first order ternary differential equations, and prove completion of pursuit in the game. Both integral constraints and geometric constraints are subjected on the players' control functions, thus two separate cases of pursuit games are examined. In the game, the pursuer wants to take the state of the system into the origin of l_2 space at some finite time interval, whereas evader avoids this from happening. For every case, we solve the control problem by establishing the admissible control function. In order to achieve the pursuer's objective, we then construct an admissible strategy for the pursuer and develop an equation for the guaranteed pursuit time of the game.

Keywords: ternary differential equations; Hilbert space; control; pursuit; strategy.

1 Introduction

The study of game theory benefited mankind in addressing problems between two parties of opposing interests. The solution to the problem entails the establishment of a feasible winning strategy for the party seeking to achieve its objective. This includes scenarios in many prominent fields such as developing an optimal control in a prey-predator model consisting of an infected prey population [35]; constructing chasing strategies for competitive control between two life insurance companies [9]; presenting an optimised pricing model for competing hotels based on energy-saving and environmental protection [25]; proposing an optimal solution for social planners in economics with a discrete control model and a unique meta-heuristic approach [10]; examining concerning trends in fisheries at East China Sea modelled as a congested marine environment [27]; analysing various capture situations and control-based techniques for unmanned aerial vehicles [11]; illustrating solution for a dynamic bi-criteria bioresource management problem [29]; and many more.

The introduction of theory of differential games by Rufus Isaacs [17] is an extension to the existing game theory, where the dynamics of the involved parties are governed by some differential equations. Given the significant amount of work that has been invested into solving differential games, numerous researchers have addressed two common constraints, that are geometric constraints and integral constraints, which are subjected on the players' control functions using a variety of methodologies. The following works can be examined for examples of differential games with (i) geometric constraints: pursuit-evasion games where one evader plays against one pursuer [31] and many pursuers [1]; a pursuit game of chasing a single faster evader [8]; a pursuit game in a nonempty closed convex set [18]; a pursuit-evasion game of finding an optimal strategy for the evader [22]; and two separate linear discrete pursuit games [28], and (ii) integral constraints: a pursuit game of determining some guaranteed pursuit time [4]; a game problem associated with a non-linear control system [12]; an optimal pursuit game set in a plane [20]; an encounter-evasion game problem [26]; and a group pursuit differential game [30].

Real life problems are often complicated and presented as a system of partial differential equations (PDEs). Thus, many initial investigations were dedicated to examine control problems and differential game problems with players' motions described by partial differential equations. Some of them are the study of a control system with PDEs of parabolic type [5]; the determination of values for such pursuit-evasion games [6]; the development of solution for game's system of fully nonlinear first-order PDEs [7]; and the verification of a unique solution by considering the notion of a generalised solution [24].

Due to the complexity of partial differential equations, the method of decomposition is used to reduce the system into an infinite system of ordinary differential equations. The decomposition method allows the differential games which were originally in the form of partial differential equations, such as a pursuit game of parabolic PDEs ([2, 23]); a pursuit game with respect to evolutionary PDEs [3]; the pursuit and evasion games under a system of PDEs with an elliptic operator [33]; a pursuit-evasion game based on a first-order evolution PDEs [34]; a control problem with the presence of another state [36]; and a pursuit game set in the entire space [38]; to be solved in the framework of an infinite system of ordinary differential equations. The article of Satimov and Tukhtasinov [32] is one of the publications that investigated a differential game problem with linear partial differential equations given by

$$z_t = Az - u + v, \quad Az = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x) z_{x_j}), \tag{1}$$

on time interval $[0, T]$. By applying the Fourier series in the decomposition method, the system that comprises of $z = z(x, t)$ is decomposed to the following ordinary differential equations:

$$\dot{z}_i(t) = \lambda_i z_i(t) - u_i(t) + v_i(t), \quad z_i(0) = z_{i0}, \tag{2}$$

for $i = 1, 2, ...,$ where $z_i, u_i, v_i \in \mathbb{R}$ and λ_i are negative eigenvalues of operater A that is as $i \to \infty, \lambda_i \to -\infty$. Pursuit and evasion games were then solved based on the reduced system. Every combination of geometric and integral constraints are imposed on the players' control functions. They pointed out sets of initial positions that are required in proving the completion of pursuit game and evasion is possible in evasion game separately.

Since then, numerous publications have devoted to study the differential game of pursuit or evasion described by an infinite system of ordinary differential equations in a structure independent from a system of partial differential equations. It should be emphasised that each game model of an infinite system is distinct from the other, meaning that solution to a game problem based on that system must be designed in a way that fits in with the system. The work of Ja'afaru and Ibragimov [19] studied a one pursuer-one evader pursuit game governed by a system of ordinary differential equations

$$\dot{z}_k(t) + \lambda_k(t) z_k(t) = -u_k(t) + v_k(t), \quad z_k(0) = z_{k0}, \tag{3}$$

where $\lambda_k(t)$ is a positive, continuous and bounded function on the time interval $[0, T]$ for $k = 1, 2, ...$ Geometric and integral constraints are considered in the game, leading to four cases of pursuit and evasion problems. The authors figured out the sufficient conditions, which also consist of set of initial positions, for every case. The establishment of admissible strategies for the players were also done.

The paper of Ibragimov et al. [15] investigated a pursuit game in the form of

$$\dot{z}_k = -\lambda_k z_k + u_k - v_k, \quad z_k(0) = z_{k0}, \quad k = 1, 2, ..., \tag{4}$$

where z_k, u_k, v_k are real numbers, with respect to geometric constraints. The coefficient $\lambda_k, k = 1, 2, ...,$ are positive real numbers. The evader's maximum speed is lesser than that of pursuer. To prove the completion of pursuit, a pursuer's strategy was constructed, and a guaranteed pursuit time was obtained. The researcher also build an admissible evader's strategy in a separate game, and guaranteed evasion time was found.

The article of Waziri et al. [39] examined a differential game of infinite one-system of simple motion by considering both geometric and integral constraints on the control functions of the players. The game occurs in a plane, and when the states of the pursuer and the evader coincide, the pursuit is considered to be completed. An admissible strategy of the pursuer was constructed. The proof was also supported with a numerical example.

The decomposition method produces a deeper understanding for the dynamic of the system, as a more complex system of partial differential equations results in reduction to a higher level of an infinite system of ordinary differential equations. Therefore, the differential game theorists later recognised a differential game could also occur in a 2-systems of differential equations. The work of Ibragimov [13] studied an optimal pursuit problem defined by the following 2-system of differential equations:

$$\begin{aligned} \dot{x}_i &= -\alpha_i x_i - \beta_i y_i + u_{i1} - v_{i1}, \qquad x_i(0) = x_{i0}, \\ \dot{y}_i &= \beta_k x_i - \alpha_i y_i + u_{i2} - v_{i2}, \qquad y_i(0) = y_{i0}, \quad i = 1, 2, ..., \end{aligned} \tag{5}$$

where $\alpha_i \geq 0, \beta_i \in \mathbb{R}$, as the equation of the game. Integral constraints is chosen to restrict the resources of the players. The solution for a time-optimal control problem was initially found.

Then, sufficient conditions were met for the termination of pursuit game and optimal strategies of the players, in attaining optimal pursuit time $\theta > 0$, were constructed.

Ibragimov et al. [14] examined a pursuit game with system (5), but with geometric constraints. They presented an equation that defines the guaranteed pursuit time of the game and determined the sufficient conditions in completing the pursuit. For the purpose of completing the pursuit, an admissible strategy for the pursuer was developed. The paper Tukhtasinov et al. [37] extended the work of [14] where the researchers established the estimated guaranteed pursuit time of the game, together with guaranteed evasion time for the evasion game.

The work of Ibragimov et al. [16] solved an optimal pursuit differential game in an infinite 2-system of differential equations different from system (5). The players' optimal strategies that satisfy integral constraints were build and optimal pursuit time was obtained.

In this present paper, we have extended the study of pursuit game to a higher level of system by designing a game model of an infinite system of first order ternary differential equations. The game is set in Hilbert space l_2 that is

$$l_2 = \{\zeta = (\zeta_1, \zeta_2, ...) : \sum_{k=1}^{\infty} |\zeta_k|^2 < \infty\},$$

with inner product and norm given by

$$< \zeta, \eta >= \sum_{k=1}^{\infty} \zeta_k \eta_k < \infty, \quad ||\zeta|| = \sqrt{< \zeta, \zeta >} = \sqrt{\sum_{k=1}^{\infty} \zeta_k^2} < \infty,$$

respectively. The proposed model is structured as ternary differential equations, describing the motion of a point in the infinite-dimensional space l_2. Unlike previous works that are limited to two-systems of differential equations models, this model is a three-systems framework. An infinite system of higher level offers a more information of the dynamics of the moving point.

We impose both integral constraints and geometric constraints on the players' control functions separately. In every case, we first solve the control problem of the system. We then develop an admissible strategy of the pursuer to complete the pursuit. Sufficient conditions are presented as well.

2 Preliminaries

We utilize the following game model:

$$\begin{aligned} \dot{x}_k &= -\alpha_k x_k - u_{k1} + v_{k1}, \quad x_k(0) = x_k^0, \\ \dot{y}_k &= -\beta_k y_k - \gamma_k z_k - u_{k2} + v_{k2}, \quad y_k(0) = y_k^0, \\ \dot{z}_k &= \gamma_k y_k - \beta_k z_k - u_{k3} + v_{k3}, \quad z_k(0) = z_k^0, \end{aligned} \tag{6}$$

where $\alpha_k, \beta_k \geq 0, \gamma_k \in \mathbb{R}$ and $u_{kj}, v_{kj} \in \mathbb{R}$ for $k = 1, 2, ...,$ and $j = 1, 2, 3$ with $x^0 = (x_1^0, x_2^0, x_3^0, ...)$, $y^0 = (y_1^0, y_2^0, y_3^0, ...)$, $z^0 = (z_1^0, z_2^0, z_3^0, ...) \in l_2$. The control parameter

$$u = (u_1, u_2, ...) = (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, ...) \in l_2,$$

of pursuer and control parameter

$$v = (v_1, v_2, ...) = (v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{23}, ...) \in l_2,$$

of evader control the motion of the state of the game.

We denote that

$$\begin{aligned} &\mu(t) = (\mu_1(t), \mu_2(t), ...) = (x_1(t), y_1(t), z_1(t), x_2(t), y_2(t), z_2(t), ...), \\ &\|\mu(t)\| = \sqrt{\sum_{k=1}^{\infty} \left(x_k^2(t) + y_k^2(t) + z_k^2(t) \right)}, \\ &\mu_k(t) = (x_k(t), y_k(t), z_k(t)), \quad |\mu_k(t)| = \sqrt{{x_k}^2(t) + {y_k}^2(t) + {z_k}^2(t)}, \\ &\mu^0 = (\mu_1^0, \mu_2^0, ...) = (x_1^0, y_1^0, z_1^0, x_2^0, y_2^0, z_2^0, ...), \\ &\|\mu^0\| = \sqrt{\sum_{k=1}^{\infty} \left((x_k^0)^2 + (y_k^0)^2 + (z_k^0)^2 \right)}, \\ &\mu_k^0 = (x_k^0, y_k^0, z_k^0), \quad |\mu_k^0| = \sqrt{(x_k^0)^2 + (y_k^0)^2 + (z_k^0)^2}. \end{aligned} \tag{7}$$

The ideas of establishment of strategies for pursuit differential game, described by the formulated model, in the present paper can be applied in various other domains such as military tactics, missile defense systems, cybersecurity, submarine movement, drone operations and so on.

The work of Madhavan et al. [21] demonstrated that if $\lambda_k \geq 0, \lambda_k = \min\{\alpha_k, \beta_k\}$ and $w(\cdot) \in S(\rho_0)$, then the following ternary differential equations:

$$\begin{aligned} \dot{x}_k &= -\alpha_k x_k + w_{k1}, \quad x_k(0) = x_k^0, \\ \dot{y}_k &= -\beta_k y_k - \gamma_k z_k + w_{k2}, \quad y_k(0) = y_k^0, \\ \dot{z}_k &= \gamma_k y_k - \beta_k z_k + w_{k3}, \quad z_k(0) = z_k^0, \end{aligned} \tag{8}$$

has a solution $\mu(\cdot) = (\mu_1(\cdot), \mu_2(\cdot), ...)$ of the form

$$\mu_k(t) = \varphi_k(t)\mu_k^0 + \int_0^t \varphi_k(t-s) w_k(s) ds, \quad k = 1, 2, ..., \tag{9}$$

where

$$\varphi_k(t) = \begin{pmatrix} e^{-\alpha_k t} & 0 & 0 \\ 0 & e^{-\beta_k t} \cos \gamma_k t & -e^{-\beta_k t} \sin \gamma_k t \\ 0 & e^{-\beta_k t} \sin \gamma_k t & e^{-\beta_k t} \cos \gamma_k t \end{pmatrix}, \tag{10}$$

for each k, which is unique and belongs to the space of continuous function in l_2 space on time interval $[0, \theta]$.

The function $\varphi_k(t), k = 1, 2, ...,$ satisfies the fact that

i. $\varphi_k^{-1}(t) = \varphi_k(-t)$,

ii. $\varphi_k(t-s) = \varphi_k(t)\varphi_k(-s) = \varphi_k(-s)\varphi_k(t)$,

iii. $|\varphi_k(t)\mu_k| \leq e^{-\lambda_k t}|\mu_k|$ where $\lambda_k = \min\{\alpha_k, \beta_k\}$.

Definition 2.1. *A function $w(\cdot) = (w_1(\cdot), w_2(\cdot), ...) \in l_2$, $w : [0, \theta] \to l_2$ which has measurable components $w_k = (w_{k1}, w_{k2}, w_{k3}), k = 1, 2, ...,$ and is called admissible control if it obeys either integral constraints*

$$\sqrt{\sum_{k=1}^{\infty} \int_0^{\theta} \sum_{j=1}^{3} |w_{kj}(t)|^2 dt} \le \rho_0, \tag{11}$$

or geometric constraints

$$\sqrt{\sum_{k=1}^{\infty} \sum_{j=1}^{3} |w_{kj}(t)|^2} \le \rho_0, t \in [0, \theta], \tag{12}$$

respectively where $\rho_0 > 0$. The set of all admissible controls based on integral constraints and geometric constraints is denoted by $S(\rho_0)$ and $S_1(\rho_0)$ respectively.

Definition 2.2. *The control function $u(\cdot) = (u_1(\cdot), u_2(\cdot), ...)$ of pursuer and the control function $v(\cdot) = (v_1(\cdot), v_2(\cdot), ...)$ of evader are admissible if they obey integral constraints*

$$\sum_{k=1}^{\infty} \int_0^{\theta} \sum_{j=1}^{3} |u_{kj}(t)|^2 dt \le \rho^2, \quad \sum_{k=1}^{\infty} \int_0^{\theta} \sum_{j=1}^{3} |v_{kj}(t)|^2 dt \le \sigma^2, \tag{13}$$

or geometric constraints

$$\sum_{k=1}^{\infty} \sum_{j=1}^{3} |u_{kj}(t)|^2 \le \rho^2, t \in [0, \theta], \quad \sum_{k=1}^{\infty} \sum_{j=1}^{3} |v_{kj}(t)|^2 \le \sigma^2, t \in [0, \theta], \tag{14}$$

where $\rho, \sigma > 0$ and $\rho > \sigma$. Set of admissible controls of pursuer (of evader) based on integral constraints and geometric constraints is denoted by $S(\rho)$ $(S(\sigma))$ and $S_1(\rho)$ $(S_1(\sigma))$ respectively.

Definition 2.3. *The function $U(\cdot, v) = (U_1(t, v), U_2(t, v), ...)$, $U : [0, \theta] \times l_2 \to l_2$ which has the form $U_k(t, v_k(t)) = v_k(t) - w_k(t), k = 1, 2, ...,$ where v_k, w_k are measurable, is called admissible strategy of the pursuer if it obeys either*

$$\sqrt{\sum_{k=1}^{\infty} \int_0^{\theta} |U_k(t, v_k(t))|^2 dt} \le \rho, \quad w(\cdot) \in S(\rho - \sigma), v(\cdot) \in S(\sigma), \tag{15}$$

or

$$\sqrt{\sum_{k=1}^{\infty} |U_k(t, v_k(t))|^2} \le \rho, \quad w(\cdot) \in S_1(\rho - \sigma), v(\cdot) \in S_1(\sigma). \tag{16}$$

Definition 2.4. *The time $\theta > 0$ is referred as guaranteed pursuit time in game (6) if there exists a strategy of pursuer U such that for any admissible control of evader, $\mu(\tau) = 0$ at some time τ, $\tau \in [0, \theta]$.*

The problem of this work is to find

1. sufficient conditions to solve control problem related to the system (8),
2. sufficient conditions to solve pursuit differential game (6),

with respect to integral constraints and geometric constraints separately.

3 Main Results

In pursuit differential games, the corresponding control problem is studied beforehand so that the constructed control can act as a reference for developing the appropriate strategy for the pursuer to successfully complete the pursuit.

3.1 Control problem

In this subsection, we are up to examine the control problem for system described by differential equations (8). We want to determine an admissible control function $w(t)$ that can steer the state of system (8) into space's origin for some time on time interval $[0, \theta]$.

Consider the following matrix

$$A_k(t) = \int_0^t \varphi_k(-s)\varphi_k^*(-s)ds = \begin{pmatrix} \int_0^t e^{2\alpha_k s}ds & 0 & 0 \\ 0 & \int_0^t e^{2\beta_k s}ds & 0 \\ 0 & 0 & \int_0^t e^{2\beta_k s}ds \end{pmatrix}, \quad k = 1, 2, ..., \tag{17}$$

and thus, clearly

$$A_k^{-1}(t) = \begin{pmatrix} \left(\int_0^t e^{2\alpha_k s}ds\right)^{-1} & 0 & 0 \\ 0 & \left(\int_0^t e^{2\beta_k s}ds\right)^{-1} & 0 \\ 0 & 0 & \left(\int_0^t e^{2\beta_k s}ds\right)^{-1} \end{pmatrix}, \tag{18}$$

where $*$ refers the transpose of the matrix.

We denote

$$B_k(t) = \begin{pmatrix} e^{2\alpha_k t} & 0 & 0 \\ 0 & e^{2\beta_k t} & 0 \\ 0 & 0 & e^{2\beta_k t} \end{pmatrix}. \tag{19}$$

We consider the following equations

$$\sum_{k=1}^{\infty} (\mu_k^0)^* A_k^{-1}(t)\mu_k^0 = \rho_0^2, \tag{20}$$

$$\sum_{k=1}^{\infty} (\mu_k^0)^* B_k(t) A_k^{-2}(t)\mu_k^0 = \rho_0^2, \tag{21}$$

where $\rho_0 > 0$.

We first implement integral constraints (13) on the players' control functions.

Theorem 3.1. *If* (20) *has a root* $t = \theta_1$*, then there exists an admissible control function* $w(\cdot) \in S(\rho_0)$ *that can steer the state of the system* (8) *to the origin at the time* θ_1*.*

Proof. We define the control

$$w_k(t) = \begin{cases} -\varphi_k^*(-t)A_k^{-1}(\theta_1)\mu_k^0, & 0 \le t \le \theta_1, \\ 0, & t > \theta_1. \end{cases} \tag{22}$$

Prove the admissibility of (22),

$$\sum_{k=1}^{\infty}\int_0^{\theta_1}|w_k(s)|^2 ds = \sum_{k=1}^{\infty}\int_0^{\theta_1}\left|-\varphi_k^*(-s)A_k^{-1}(\theta_1)\mu_k^0\right|^2 ds = \sum_{k=1}^{\infty}\int_0^{\theta_1}\left|\varphi_k^*(-s)A_k^{-1}(\theta_1)\mu_k^0\right|^2 ds.$$

Observe that

$$\varphi_k^*(-s)A_k^{-1}(\theta_1)\mu_k^0 = \begin{pmatrix} x_k^0 e^{\alpha_k s}\Big(\int_0^{\theta_1} e^{2\alpha_k t}dt\Big)^{-1} \\ y_k^0 e^{\beta_k s}\cos\gamma_k s\Big(\int_0^{\theta_1} e^{2\beta_k t}dt\Big)^{-1} - z_k^0 e^{\beta_k s}\sin\gamma_k s\Big(\int_0^{\theta_1} e^{2\beta_k t}dt\Big)^{-1} \\ y_k^0 e^{\beta_k s}\sin\gamma_k s\Big(\int_0^{\theta_1} e^{2\beta_k t}dt\Big)^{-1} + z_k^0 e^{\beta_k s}\cos\gamma_k s\Big(\int_0^{\theta_1} e^{2\beta_k t}dt\Big)^{-1} \end{pmatrix}.$$

By definition of θ_1, we now obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} & \int_0^{\theta_1}|w_k(s)|^2 ds \\
&= \sum_{k=1}^{\infty}\int_0^{\theta_1}\left((x_k^0)^2 e^{2\alpha_k s}\left(\int_0^{\theta_1} e^{2\alpha_k t}dt\right)^{-2} + (y_k^0)^2 e^{2\beta_k s}\left(\int_0^{\theta_1} e^{2\beta_k t}dt\right)^{-2}\right. \\
&\qquad \left. + (z_k^0)^2 e^{2\beta_k s}\left(\int_0^{\theta_1} e^{2\beta_k t}dt\right)^{-2}\right)ds \\
&= \sum_{k=1}^{\infty}\left((x_k^0)^2\left(\int_0^{\theta_1} e^{2\alpha_k t}dt\right)^{-1} + (y_k^0)^2\left(\int_0^{\theta_1} e^{2\beta_k t}dt\right)^{-1} + (z_k^0)^2\left(\int_0^{\theta_1} e^{2\beta_k t}dt\right)^{-1}\right) \\
&= \sum_{k=1}^{\infty}(\mu_k^0)^* A_k^{-1}(\theta_1)\mu_k^0 = \rho_0^2.
\end{aligned}$$

This indicates that (22) satisfies integral constraints (13).

Now, we show that $\mu(\theta_1) = 0$. Substituting constructed control (22) into (9), we have that

$$\begin{aligned}
\mu_k(\theta_1) &= \varphi_k(\theta_1)\left(\mu_k^0 + \int_0^{\theta_1}\varphi_k(-s)\big(-\varphi_k^*(-s)\varphi_k^{-1}(\theta_1)\mu_k^0\big)ds\right) \\
&= \varphi_k(\theta_1)\left(\mu_k^0 - \int_0^{\theta_1}\varphi_k(-s)\varphi_k^*(-s)ds A_k^{-1}(\theta_1)\mu_k^0\right),
\end{aligned}$$

by (17),

$$\begin{aligned}
\mu_k(\theta_1) &= \varphi_k(\theta_1)\Big(\mu_k^0 - A_k(\theta_1)A_k^{-1}(\theta_1)\mu_k^0\Big) \\
&= \varphi_k(\theta_1)(\mu_k^0 - \mu_k^0) = 0.
\end{aligned}$$

Hence, the state of system (6) is steered into origin at finite time θ_1. This proves the theorem. □

Next, let us consider the case where the control functions of the players satisfy geometric constraints (14).

Theorem 3.2. *If (21) has a root $t = \theta_2$, then there exists an admissible control function $w(\cdot) \in S_1(\rho_0)$ that can steer the state of the system (8) to the origin at the time θ_2.*

Proof. We build the control

$$w_k(t) = \begin{cases} -\varphi_k^*(-t)A_k^{-1}(\theta_2)\mu_k^0, & 0 \le t \le \theta_2, \\ 0, & t > \theta_2. \end{cases} \tag{23}$$

Despite having a similar form, the distinction of control functions (22) and (23) becomes evident in the proof of their admissibility, as every constructed control function must adhere to its respective conditions (20) and (21).

Here, we show that (23) satisfies geometric constraints. We have

$$\begin{aligned}\sum_{k=1}^{\infty} |w_k(t)|^2 = \sum_{k=1}^{\infty} \Bigg(&(x_k^0)^2 e^{2\alpha_k t} \left(\int_0^{\theta_2} e^{2\alpha_k s} ds \right)^{-2} + (y_k^0)^2 e^{2\beta_k t} \Big(\int_0^{\theta_2} e^{2\beta_k s} ds \Big)^{-2} \\ &+ (z_k^0)^2 e^{2\beta_k t} \left(\int_0^{\theta_2} e^{2\beta_k s} ds \right)^{-2} \Bigg).\end{aligned}$$

Since $t \le \theta_2$,

$$\begin{aligned}\sum_{k=1}^{\infty} |w_k(t)|^2 &\le \sum_{k=1}^{\infty} (x_k^0)^2 e^{2\alpha_k \theta_2} \Big(\int_0^{\theta_2} e^{2\alpha_k s} ds \Big)^{-2} + (y_k^0)^2 e^{2\beta_k \theta_2} \left(\int_0^{\theta_2} e^{2\beta_k s} ds \right)^{-2} \\ &\quad + (z_k^0)^2 e^{2\beta_k \theta_2} \left(\int_0^{\theta_2} e^{2\beta_k s} ds \right)^{-2} \\ &= \sum_{k=1}^{\infty} (\mu_k^0)^* B_k(\theta_2) A_k^{-2}(\theta_2)\mu_k^0 = \rho_0^2.\end{aligned}$$

Thus, control $w(\cdot)$ (23) belongs to $S_1(\rho_0)$.

We use (23) to prove that $\mu(\theta_2) = 0$ as follows;

$$\begin{aligned}\mu_k(\theta_2) &= \varphi_k(\theta_2) \left(\mu_k^0 + \int_0^{\theta_2} \varphi_k(-s)\big(- \varphi_k^*(-s)A_k^{-1}(\theta_2)\mu_k^0\big) ds \right) \\ &= \varphi_k(\theta_2) \left(\mu_k^0 - \int_0^{\theta_2} \varphi_k(-s)\varphi_k^*(-s) ds A_k^{-1}(\theta_2)\mu_k^0 \right) \\ &= \varphi_k(\theta_2)\Big(\mu_k^0 - A_k(\theta_2)A_k^{-1}(\theta_2)\mu_k^0 \Big) \\ &= \varphi_k(\theta_2)(\mu_k^0 - \mu_k^0) = 0.\end{aligned}$$

Therefore, (23) is able to bring the system's state into origin of l_2 at finite time θ_2. This completes the proof. □

3.2 Differential game of pursuit

Now, we examine pursuit differential game defined by (6), which describes the motion of a point in space l_2 which is controlled by the control functions $u(\cdot)$ and $v(\cdot)$. In this subsection, our target is to achieve pursuer's objective that is to complete the pursuit by constructing strategy for the pursuer to drag the point into the origin at some time, for any admissible control of the evader. The game of pursuit is considered based on integral and geometric constraints in separate cases.

For every $k = 1, 2, ...$, the solution $\mu_k(t), k = 1, 2, ...$, is expressed as

$$\mu_k(t) = \varphi_k(t)\mu_k^0 + \int_0^t \varphi_k(t-s)(-U_k(s, v_k(s)) + v_k(s))ds. \tag{24}$$

Here, we consider, for every k,

$$\sum_{k=1}^{\infty}(\mu_k^0)^* A_k^{-1}(t)\mu_k^0 = (\rho - \sigma)^2, \tag{25}$$

$$\sum_{k=1}^{\infty}(\mu_k^0)^* B_k(t) A_k^{-2}(t)\mu_k^0 = (\rho - \sigma)^2, \tag{26}$$

where $\rho > \sigma$.

In the first case, we solve pursuit problem with integral constraints constraining the players' control functions.

Theorem 3.3. *If $\rho > \sigma$ and (25) has a root $t = \theta_3$, then the game of pursuit (6) with (13) is completed at the time θ_3.*

Proof. We provide the pursuer with strategy given by

$$U_k(t, v_k(t)) = \begin{cases} \varphi_k^*(-t)A_k^{-1}(\theta_3)\mu_k^0 + v_k(t), & 0 \le t \le \theta_3, \\ 0, & t > \theta_3. \end{cases} \tag{27}$$

We guarantee the admissibility of (27) where $v(\cdot) \in S(\sigma)$. Using Minowskii's inequality, we obtain that

$$\begin{aligned}\sqrt{\sum_{k=1}^{\infty}\int_0^{\theta_3} |U_k(s, v_k(s))|^2 ds} &\le \sqrt{\sum_{k=1}^{\infty}\int_0^{\theta_3} \left|\varphi_k^*(-s)A_k^{-1}(\theta_3)\mu_k^0\right|^2 ds} + \sqrt{\sum_{k=1}^{\infty}\int_0^{\theta_3} |v_k(s)|^2 ds} \\ &\le \sqrt{\sum_{k=1}^{\infty}\int_0^{\theta_3} \left|\varphi_k^*(-s)A_k^{-1}(\theta_3)\mu_k^0\right|^2 ds} + \sigma.\end{aligned}$$

We then get that

$$\begin{aligned}
&\sqrt{\sum_{k=1}^{\infty}\int_0^{\theta_3}|U_k(s,v_k(s))|^2ds}\\
&=\sqrt{\sum_{k=1}^{\infty}\bigg((x_k^0)^2\Big(\int_0^{\theta_3}e^{2\alpha_k t}dt\Big)^{-1}+(y_k^0)^2\Big(\int_0^{\theta_3}e^{2\beta_k t}dt\Big)^{-1}+(z_k^0)^2\Big(\int_0^{\theta_3}e^{2\beta_k t}dt\Big)^{-1}\bigg)}+\sigma\\
&=\sqrt{\sum_{k=1}^{\infty}(\mu_k^0)^*A_k^{-1}(\theta_3)\mu_k^0}+\sigma\\
&=\sqrt{(\rho-\sigma)^2}+\sigma=\rho-\sigma+\sigma=\rho.
\end{aligned}$$

This proves that (23) satisfies integral constraints.

We show that the pursuer can use strategy (27) to complete the pursuit. By substituting into (24), we have that

$$\begin{aligned}
\mu_k(\theta_3)=&\varphi_k(\theta_3)\bigg(\mu_k^0+\int_0^{\theta_3}\varphi_k(-s)\big(-\varphi_k^*(-s)A_k^{-1}(\theta_3)\mu_k^0-v_k(s)\big)ds+\int_0^{\theta_3}\varphi_k(-s)v_k(s)ds\bigg)\\
=&\varphi_k(\theta_3)\bigg(\mu_k^0-\int_0^{\theta_3}\varphi_k(-s)\varphi_k^*(-s)dsA_k^{-1}(\theta_3)\mu_k^0-\int_0^{\theta_3}\varphi_k(-s)v_k(s)ds\\
&+\int_0^{\theta_3}\varphi_k(-s)v_k(s)ds\bigg)\\
=&\varphi_k(\theta_3)\Big(\mu_k^0-A_k(\theta_3)A_k^{-1}(\theta_3)\mu_k^0\Big)\\
=&\varphi_k(\theta_3)(\mu_k^0-\mu_k^0)=0.
\end{aligned}$$

Hence, the pursuit can be completed at time θ_3. This finishes the proof. □

Meanwhile, in the second case, we impose geometric constraints on players' control functions.

Theorem 3.4. *If $\rho>\sigma$ and (26) has a root $t=\theta_4$, then the game of pursuit (6) with (14) is completed at finite time θ_4.*

Proof. We present the following strategy to the pursuer

$$U_k(t,v_k(t))=\begin{cases}\varphi_k^*(-t)A_k^{-1}(\theta_4)\mu_k^0+v_k(t), & 0\le t\le\theta_4,\\ 0, & t>\theta_4.\end{cases}\tag{28}$$

To show that (28) is admissible, we employ Minowskii's inequality and the fact that $v(\cdot)\in S_1(\sigma)$

$$\begin{aligned}
\sqrt{\sum_{k=1}^{\infty}|U_k(t,v_k(t))|^2}&\le\sqrt{\sum_{k=1}^{\infty}\left|\varphi_k^*(-t)A_k^{-1}(\theta_4)\mu_k^0\right|^2}+\sqrt{\sum_{k=1}^{\infty}|v_k(t)|^2}\\
&\le\sqrt{\sum_{k=1}^{\infty}\left|\varphi_k^*(-t)A_k^{-1}(\theta_4)\mu_k^0\right|^2}+\sigma^2.
\end{aligned}$$

We then find that

$$\sqrt{\sum_{k=1}^{\infty}|U_k(t,v_k(t))|^2}$$
$$= \sqrt{\sum_{k=1}^{\infty}\left((x_k^0)^2e^{2\alpha_k t}\Big(\int_0^{\theta_4}e^{2\alpha_k s}ds\Big)^{-2}+(y_k^0)^2e^{2\beta_k t}\Big(\int_0^{\theta_4}e^{2\beta_k s}ds\Big)^{-2}+(z_k^0)^2e^{2\beta_k t}\Big(\int_0^{\theta_4}e^{2\beta_k s}ds\Big)^{-1}\right)}$$
$$+\sigma$$
$$\leq \sqrt{\sum_{k=1}^{\infty}\left((x_k^0)^2e^{2\alpha_k \theta_4}\Big(\int_0^{\theta_4}e^{2\alpha_k s}ds\Big)^{-2}+(y_k^0)^2e^{2\beta_k \theta_4}\Big(\int_0^{\theta_4}e^{2\beta_k s}ds\Big)^{-2}+(z_k^0)^2e^{2\beta_k \theta_4}\Big(\int_0^{\theta_4}e^{2\beta_k s}ds\Big)^{-1}\right)}$$
$$+\sigma$$
$$= \sqrt{\sum_{k=1}^{\infty}(\mu_k^0)^*B_k(\theta_4)A_k^{-2}(\theta_4)\mu_k^0}+\sigma=\sqrt{(\rho-\sigma)^2}+\sigma=\rho.$$

Show that pursuit game is terminated at time θ_4. Using admissible strategy (28), we get

$$\mu_k(\theta_4)=\varphi_k(\theta_4)\left(\mu_k^0+\int_0^{\theta_4}\varphi_k(-s)\big(-\varphi_k^*(-s)A_k^{-1}(\theta_4)\mu_k^0-v_k(s)\big)ds+\int_0^{\theta_4}\varphi_k(-s)v_k(s)ds\right)$$
$$=\varphi_k(\theta_4)\left(\mu_k^0-A_k(\theta_4)A_k^{-1}(\theta_4)\mu_k^0\right)$$
$$=\varphi_k(\theta_4)(\mu_k^0-\mu_k^0)=0.$$

This completes the proof of the theorem. □

We now provide an example to illustrate our results.

Example 3.1. *We consider a pursuit game defined by ternary differential equations (6) where the control functions of pursuer and evader satisfy (13). Let*

$$x_k^0=y_k^0=z_k^0=\frac{1}{k},\quad \alpha_k=\beta_k=\frac{1}{2},\quad \rho=2,\sigma=1, \tag{29}$$

for $k=1,2,\dots$. Then, (25) takes the form

$$\sum_{k=1}^{\infty}\frac{3}{k^2}\left(\int_0^t e^s ds\right)^{-1}=(2-1)^2,$$

which is equivalent to

$$\frac{3}{e^t-1}\sum_{k=1}^{\infty}\frac{1}{k^2}=1.$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$, therefore we obtain

$$3\left(\frac{\pi^2}{6}\right)=e^t-1,$$
$$e^t=\frac{\pi^2}{2}+1.$$

Clearly,

$$t=\ln\left(\frac{\pi^2}{2}+1\right).$$

Hence, $\theta_3=\ln\left(\frac{\pi^2}{2}+1\right)$, and by Theorem 3.3, the pursuit in the game for initial values (29) is completed at the time θ_3.

4 Conclusions

In this paper, we have investigated two cases of pursuit differential games of an infinite system of first order ternary differential equations, by considering both integral and geometric constraints. For every case, we have determined sufficient conditions that enables the state of the system steers into origin, and solved control problem related to the system. Moreover, we have established admissible strategy for the pursuer in both cases and shown the pursuit is completed at a guaranteed pursuit time.

Future research could explore evasion differential games described by the formulated model (6). To add some challenges, one could consider imposing mixed constraints on the control functions of the players to solve the differential game problem based on system (6).

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